

ON INTRINSICALLY KNOTTED OR COMPLETELY 3-LINKED GRAPHS

RYO HANAOKI, RYO NIKKUNI, KOUKI TANIYAMA, AND AKIKO YAMAZAKI

ABSTRACT. We say that a graph is intrinsically knotted or completely 3-linked if every embedding of the graph into the 3-sphere contains a nontrivial knot or a 3-component link any of whose 2-component sublink is nonsplittable. We show that a graph obtained from the complete graph on seven vertices by a finite sequence of $\triangle Y$ -exchanges and $Y\triangle$ -exchanges is a minor-minimal intrinsically knotted or completely 3-linked graph.

1. INTRODUCTION

Throughout this paper we work in the piecewise linear category. Let f be an embedding of a finite graph G into the 3-sphere. Then f is called a *spatial embedding* of G and $f(G)$ is called a *spatial graph*. We denote the set of all spatial embeddings of G by $\text{SE}(G)$. We call a subgraph γ of G which is homeomorphic to the circle a *cycle* of G . For a positive integer n , $\Gamma^{(n)}(G)$ denotes the set of all cycles of G if $n = 1$ and the set of all unions of mutually disjoint n cycles of G if $n \geq 2$. In particular, we denote $\Gamma^{(1)}(G)$ by $\Gamma(G)$ simply. For an element λ in $\Gamma^{(n)}(G)$ and a spatial embedding f of G , $f(\lambda)$ is none other than a knot if $n = 1$ and an n -component link if $n \geq 2$.

A graph G is said to be *intrinsically linked* (IL) if for every spatial embedding f of G , $f(G)$ contains a nonsplittable 2-component link. Conway-Gordon [1] and Sachs [20] showed that K_6 is IL, where K_m denotes the *complete graph* on m vertices. Moreover, IL graphs have been completely characterized as follows. For a graph G and an edge e of G , we denote the subgraph $G \setminus \text{inte}$ by $G - e$. Let $e = \overline{uv}$ is an edge of G which is not a loop. We call the graph which is obtained from $G - e$ by identifying the end vertices u and v the *edge contraction of G along e* and denote it by G/e . A graph H is called a *minor* of a graph G if there exists a subgraph G' of G and the edges e_1, e_2, \dots, e_m of G' such that H is obtained from G' by a sequence of edge contractions along e_1, e_2, \dots, e_m . A minor H of G is called a *proper minor* if H does not equal G . Let \mathcal{P} be a property for graphs which is *closed* under minor reductions; that is, for any graph G which does not have \mathcal{P} , all minors of G also do not have \mathcal{P} . A graph G is said to be *minor-minimal* with respect to \mathcal{P} if G has \mathcal{P} but all proper minors of G do not have \mathcal{P} . Note that G has \mathcal{P} if and only if G has a minor-minimal graph with respect to \mathcal{P} as a minor. By the famous theorem of Robertson-Seymour [18], there are finitely many minor-minimal graphs with respect

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to \mathcal{P} . Nešetřil-Thomas [16] showed that IL is closed under minor reductions, and Robertson-Seymour-Thomas [19] showed that the set of all minor-minimal graphs with respect to IL equals the *Petersen family* which is the set of all graphs obtained from K_6 by a finite sequence of ΔY -exchanges and $Y\Delta$ -exchanges. Here a ΔY -exchange is an operation to obtain a new graph G_Y from a graph G_Δ by removing all edges of a cycle Δ of G_Δ with exactly three edges \overline{uv} , \overline{vw} and \overline{wu} , and adding a new vertex x and connecting it to each of the vertices u, v and w as illustrated in Fig. 1.1 (we often denote $\overline{ux} \cup \overline{vx} \cup \overline{wx}$ by Y). A $Y\Delta$ -exchange is the reverse of this operation. This family contains exactly seven graphs as illustrated in Fig. 1.2, where $G \rightarrow G'$ means that G' can be obtained from G by a single ΔY -exchange. Note that P_{10} is isomorphic to the *Petersen graph*. We remark here that if G_Δ is IL then G_Y is also IL [15], and if G_Y is IL then G_Δ is also IL [19]. Namely ΔY and $Y\Delta$ -exchanges preserve IL.

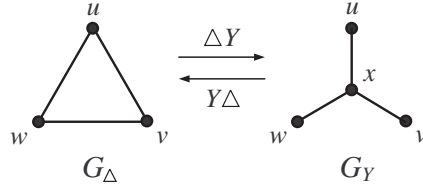


FIGURE 1.1.

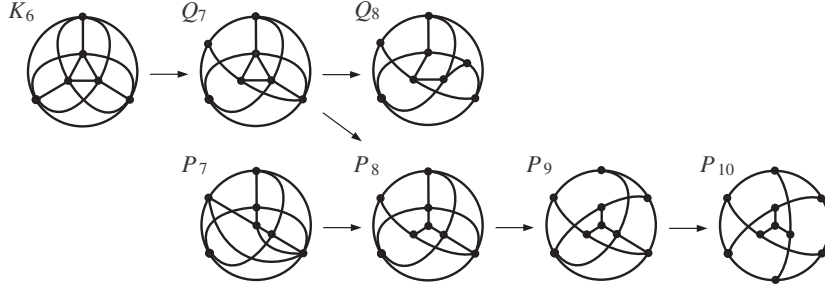


FIGURE 1.2.

On the other hand, a graph G is said to be *intrinsically knotted* (IK) if for every spatial embedding f of G , $f(G)$ contains a nontrivial knot. Conway-Gordon [1] showed that K_7 is IK. Fellows and Langston [2] showed that IK is closed under minor reductions, and Motwani-Raghunathan-Saran [15] showed that K_7 is a minor-minimal IK graph. Although additional minor-minimal IK graphs are known by Kohara-Suzuki [13] and Foisy [6], [7], IK graphs have not been completely characterized yet. We remark here that if G_Δ is IK then G_Y is also IK [15], but if G_Y is IK then G_Δ may not always be IK. Namely the $Y\Delta$ -exchange does not preserve IK in general. Actually Flapan-Naimi [3] exhibited that there exists a graph G_{FN} which is obtained from K_7 by five times of ΔY -exchanges and twice $Y\Delta$ -exchanges such that it is not IK. We call the set of all graphs obtained from K_7 by a finite

sequence of $\triangle Y$ and $Y\triangle$ -exchanges the *Heawood family*.¹ This family contains exactly twenty graphs as illustrated in Fig. 1.3, where $G \rightarrow G'$ means that G' can be obtained from G by a single $\triangle Y$ -exchange. Note that C_{14} is isomorphic to the *Heawood graph*, see Remark 4.7.

Kohara-Suzuki [13] showed that a graph G in the Heawood family is a minor-minimal IK graph if G is obtained from K_7 by a finite sequence of $\triangle Y$ -exchanges, namely G is one of fourteen graphs $K_7, H_8, H_9, \dots, H_{12}, F_9, F_{10}, E_{10}, E_{11}$ and $C_{11}, C_{12}, \dots, C_{14}$.² On the other hand, N'_{10} is isomorphic to G_{FN} , namely N'_{10} is not IK. Our first purpose in this paper is to determine completely when a graph in the Heawood family is IK as follows.

Theorem 1.1. *Let G be a graph in the Heawood family. Then the following are equivalent:*

- (1) G is IK,
- (2) G is obtained from K_7 by a finite sequence of $\triangle Y$ -exchanges,
- (3) $\Gamma^{(3)}(G)$ is the empty set.

Namely, each of the graphs $N_9, N_{10}, N_{11}, N'_{10}, N'_{11}$ and N'_{12} in the Heawood family is not IK, and only these graphs in the Heawood family contain a union of mutually disjoint three cycles. Our second purpose in this paper is to show that any of the graphs in the Heawood family is a minor-minimal graph with respect to a certain kind of intrinsic nontriviality even if it is not IK. We say that a graph G is *intrinsically knotted or completely 3-linked* (I(K or C3L)) if for every spatial embedding f of G , $f(G)$ contains a nontrivial knot or a 3-component link any of whose 2-component sublink is nonsplittable. Note that an IK graph is I(K or C3L). As we will show in Proposition 2.2, I(K or C3L) is closed under minor reductions. Then we have the the following.

Theorem 1.2. *All of the graphs in the Heawood family are minor-minimal I(K or C3L) graphs.*

Actually, each of the graphs $N_9, N_{10}, N_{11}, N'_{10}, N'_{11}$ and N'_{12} in the Heawood family is not IK but I(K or C3L), and they are minor-minimal with respect to I(K or C3L).

Remark 1.3. (1) A graph G is said to be *intrinsically n -linked* (InL) if for every spatial embedding f of G , $f(G)$ contains a nonsplittable n -component link [4] [5]. Note that I2L coincides with IL. Let G be a graph in the Heawood family which is not IK. Then we will show in Example 4.6 that there exists a spatial embedding f of G such that $f(G)$ does not contain a nonsplittable 3-component link. Namely G is neither IK nor I3L.

- (2) A graph G is said to be *intrinsically knotted or 3-linked* (I(K or 3L)) if for every spatial embedding f of G , $f(G)$ contains a nontrivial knot or a nonsplittable 3-component link [8]. It is clear that I(K or C3L) implies I(K or 3L), but the converse is not true. Actually in [8], although Foisy discovered an I(K or 3L) graph G and exhibit a spatial embedding f of G such that $f(G)$ contains a nonsplittable 3-component link but does not

¹In [10], van der Holst call the set of all graphs obtained from K_7 or $K_{3,3,1,1}$ by a finite sequence of $\triangle Y$ and $Y\triangle$ -exchanges the Heawood family, where $K_{3,3,1,1}$ is the complete 4-partite graph on $3 + 3 + 1 + 1$ vertices.

²We remark that one edge of F_{10} in [13, Fig. 5] is wanting.

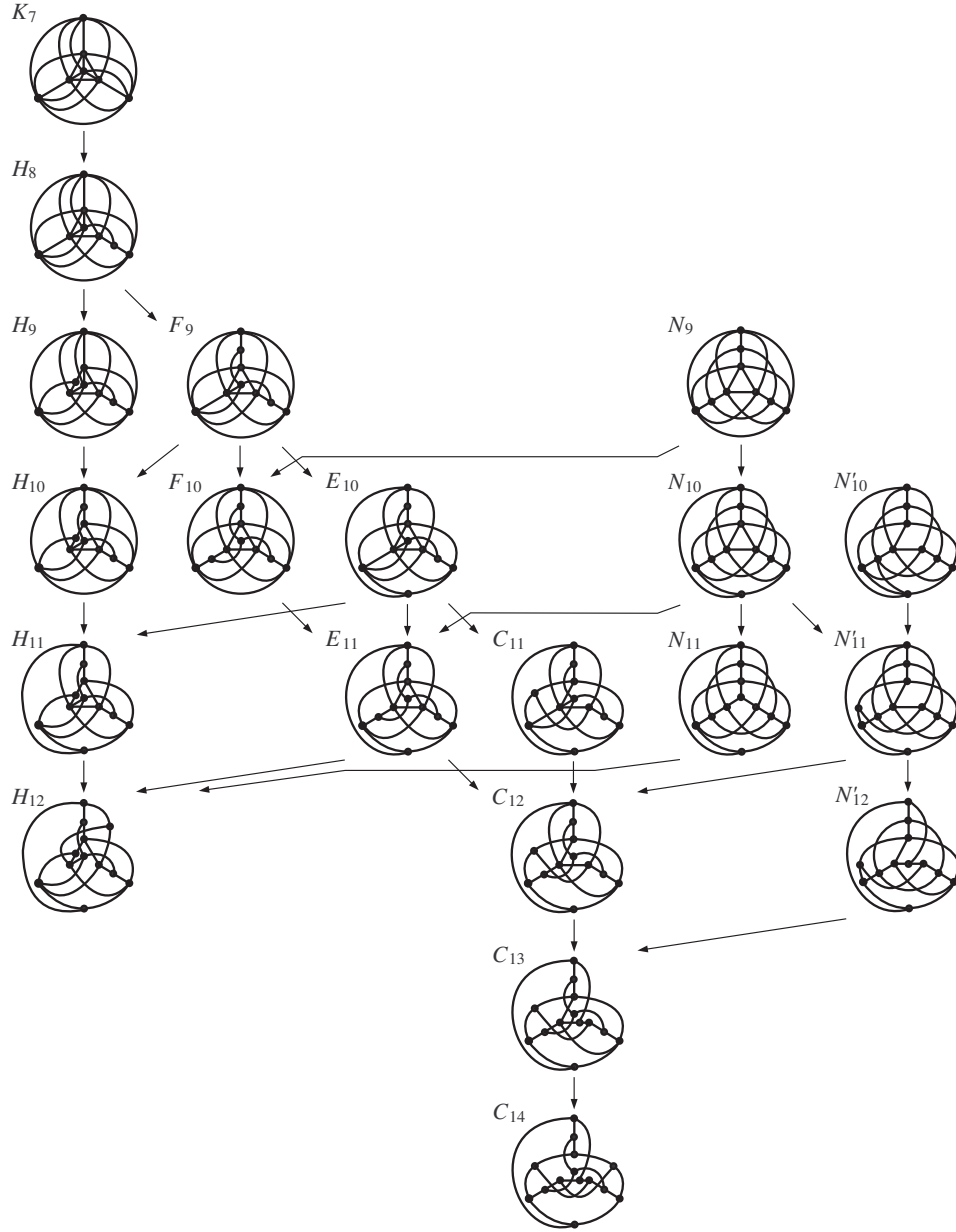


FIGURE 1.3.

contain a nontrivial knot, each of the nonsplittable 3-component links in $f(G)$ contains a split 2-component sublink.

The rest of this paper is organized as follows. In the next section, we show the general results about graph minors, ΔY -exchanges and spatial graphs. We prove Theorems 1.1 and 1.2 in sections 3 and 4, respectively.

2. GRAPH MINORS, ΔY -EXCHANGES AND SPATIAL GRAPHS

Let H be a minor of a graph G . Then there exists a natural injection

$$\Psi^{(n)} = \Psi_{H,G}^{(n)} : \Gamma^{(n)}(H) \longrightarrow \Gamma^{(n)}(G)$$

for any positive integer n . In particular, we denote $\Psi^{(1)}$ by Ψ simply. Let f be a spatial embedding of G and e an edge of G which is not a loop. Then by contracting $f(e)$ into one point, we obtain a spatial embedding $\psi(f)$ of G/e . Similarly we also can obtain a spatial embedding $\psi(f)$ of H from f . Thus we obtain a map

$$\psi = \psi_{G,H} : \text{SE}(G) \longrightarrow \text{SE}(H).$$

Then we immediately have the following.

Proposition 2.1. *For a spatial embedding f of G and an element λ in $\Gamma^{(n)}(H)$, $\psi(f)(\lambda)$ is ambient isotopic to $f(\Psi^{(n)}(\lambda))$. \square*

Now we show the following.

Proposition 2.2. *$I(K \text{ or } C3L)$ is closed under minor reductions.*

Proof. Let G be a graph which is not $I(K \text{ or } C3L)$ and H a minor of G . Let f be a spatial embedding of G which contains neither a nontrivial knot nor a 3-component link any of whose 2-component sublink is nonsplittable. Then by Proposition 2.1, $\psi(f)$ also contains neither a nontrivial knot nor a 3-component link any of whose 2-component sublink is nonsplittable. This implies that H is not $I(K \text{ or } C3L)$. \square

Remark 2.3. Proposition 2.1 also implies that IK , InL and $I(K \text{ or } 3L)$ are closed under minor reductions.

Let G_Δ and G_Y be two graphs such that G_Y is obtained from G_Δ by a single ΔY -exchange as we said in the previous section. Let λ be an element in $\Gamma^{(n)}(G_\Delta)$ which does not contain Δ . Then there exists an element $\Phi^{(n)}(\lambda)$ in $\Gamma^{(n)}(G_Y)$ such that $\lambda \setminus \Delta = \Phi^{(n)}(\lambda) \setminus Y$. Thus we obtain a map

$$\Phi^{(n)} = \Phi_{G_\Delta, G_Y}^{(n)} : \left\{ \lambda \in \Gamma^{(n)}(G_\Delta) \mid \lambda \not\supset \Delta \right\} \longrightarrow \Gamma^{(n)}(G_Y)$$

for any positive integer n . In particular, we denote $\Phi^{(1)}$ by Φ simply. Note that $\Phi^{(n)}$ is surjective and the inverse image of λ by $\Phi^{(n)}$ contains at most two elements in $\Gamma^{(n)}(G_\Delta)$ for any element λ in $\Gamma^{(n)}(G_Y)$. Note also that the surjectivity of $\Phi^{(n)}$ implies the following.

Proposition 2.4. *For $n \geq 2$, if $\Gamma^{(n)}(G_\Delta)$ is the empty set, then $\Gamma^{(n)}(G_Y)$ is also the empty set. \square*

Let f be a spatial embedding of G_Y and D a 2-disk in the 3-sphere such that $D \cap f(G_Y) = f(Y)$ and $\partial D \cap f(G_Y) = \{f(u), f(v), f(w)\}$. Let $\varphi(f)$ a spatial embedding of G_Δ such that $\varphi(f)(x) = f(x)$ for $x \in G_Y \setminus Y$ and $\varphi(f)(G_\Delta) = (f(G_Y) \setminus f(Y)) \cup \partial D$. Thus we obtain a map

$$\varphi = \varphi_{G_Y, G_\Delta} : \text{SE}(G_Y) \longrightarrow \text{SE}(G_\Delta).$$

Then we immediately have the following.

Proposition 2.5. *For a spatial embedding f of G_Y and an element λ in $\Gamma^{(n)}(G_Y)$, $f(\lambda)$ is ambient isotopic to $\varphi(f)(\lambda')$ for each element λ' in the inverse image of λ by $\Phi^{(n)}$. \square*

Now we show the following lemmas.

Lemma 2.6. *If G_Δ is $I(K$ or $C3L)$, then G_Y is also $I(K$ or $C3L)$.*

Proof. Assume that G_Y is not $I(K$ or $C3L)$, namely there exists a spatial embedding f of G_Y which contains neither a nontrivial knot nor a 3-component link any of whose 2-component sublink is nonsplittable. In the following we show that $\varphi(f)(G_\Delta)$ also contains neither a nontrivial knot nor a 3-component link any of whose 2-component sublink is nonsplittable. Let γ be an element in $\Gamma(G_\Delta)$. If γ is not Δ , then $\varphi(f)(\gamma)$ is ambient isotopic to $f(\Phi(\gamma))$ by Proposition 2.5 and $f(\Phi(\gamma))$ is a trivial knot by the assumption. Since $\varphi(f)(\Delta)$ is also a trivial knot, it follows that $\varphi(f)(G_\Delta)$ does not contain a nontrivial knot. Let λ be an element in $\Gamma^{(3)}(G_\Delta)$. If λ does not contain Δ , then $\varphi(f)(\lambda)$ is ambient isotopic to $f(\Phi^{(3)}(\lambda))$ by Proposition 2.5 and $f(\Phi^{(3)}(\lambda))$ is a 3-component link which contains a split 2-component sublink by the assumption. If λ contains Δ , then $\varphi(f)(\lambda)$ is a split 3-component link. Thus we see that $\varphi(f)(G_\Delta)$ does not contain a 3-component link any of whose 2-component sublink is nonsplittable. \square

Lemma 2.7. *If G_Y is minor-minimal for $I(K$ or $C3L)$, then G_Δ is also minor-minimal for $I(K$ or $C3L)$.*

Proof. In the following we show that for any edge e of G_Δ which is not a loop, there exists a spatial embedding f of $G_\Delta - e$ and a spatial embedding g of G_Δ/e such that each of $f(G_\Delta - e)$ and $g(G_\Delta/e)$ contains neither a nontrivial knot nor a 3-component link any of whose 2-component sublink is nonsplittable. If e is not \overline{uv} , \overline{vw} or \overline{wu} , then there exists a spatial embedding f' of $G_Y - e$ and a spatial embedding g' of G_Y/e such that both $f'(G_Y - e)$ and $g'(G_Y/e)$ contains neither a nontrivial knot nor a 3-component link any of whose 2-component sublink is nonsplittable. Note that $G_Y - e$ (resp. G_Y/e) is obtained from $G_\Delta - e$ (resp. G_Δ/e) by a single ΔY -exchange at the same Δ . Then we see that each of $\varphi(f')(G_\Delta - e)$ and $\varphi(g')(G_\Delta/e)$ contains neither a nontrivial knot nor a 3-component link any of whose 2-component sublink is nonsplittable in the similar way as the proof of Lemma 2.6. If e is one of \overline{uv} , \overline{vw} and \overline{wu} , we may assume that $e = \overline{uv}$ without loss of generality. Now there exists a spatial embedding f' of G_Y/\overline{xw} such that $f'(G_Y/\overline{xw})$ contains neither a nontrivial knot nor a 3-component link any of whose 2-component sublink is nonsplittable. Then we can see that $G_\Delta - \overline{uv} = G_Y/\overline{xw}$. On the other hand, there exists a spatial embedding g' of $G_Y/\overline{xv}/\overline{xu}$ such that $g'(G_Y/\overline{xv}/\overline{xu})$ contains neither a nontrivial knot nor a 3-component link any of whose 2-component sublink is nonsplittable. Take a 2-disk D' in the 3-sphere such that $D' \cap g'(G_Y/\overline{xv}/\overline{xu}) = g'(\overline{uw})$ and $\partial D' \cap g'(G_Y/\overline{xv}/\overline{xu}) = \{g'(u), g'(w)\}$. Then $(g'(G_Y/\overline{xv}/\overline{xu}) \setminus \text{int} g'(\overline{uw})) \cup \partial D'$ may be regarded as the image of a spatial embedding of G_Δ/\overline{uv} , which is denoted by g . It is clear that $g(G_\Delta/\overline{uv})$ contains neither a nontrivial knot nor a 3-component link any of whose 2-component sublink is nonsplittable. \square

Remark 2.8. Lemma 2.7 has already been proven by Ozawa-Tsutsumi in a more general form [17, Lemma 3.1, Exercise 3.2]. But we give a proof of Lemma 2.7 as described above for the reader's convenience.

3. PROOF OF THEOREM 1.1

Lemma 3.1. *Each of the graphs $N_9, N_{10}, N_{11}, N'_{10}, N'_{11}$ and N'_{12} in the Heawood family is not IK.*

Proof. Since the case of N'_{10} has been already shown by Flapan-Naimi [3], we show that each of the graphs $N_9, N_{10}, N_{11}, N'_{11}$ and N'_{12} is not IK. Let f_9 be the spatial embedding of N_9 as illustrated in Fig. 3.1. Then it can be checked directly that $f_9(N_9)$ does not contain a nontrivial knot. Thus N_9 is not IK. Let f_{10} be the spatial embedding of N_{10} as illustrated in Fig. 3.1. Let φ_{N_{10}, N_9} be the map from $\text{SE}(N_{10})$ to $\text{SE}(N_9)$ induced by the $Y\Delta$ -exchange from N_{10} to N_9 at the marked Y as illustrated in Fig. 3.1. Then it is clear that $\varphi(f_{10}) = f_9$. Since $f_9(N_9)$ does not contain a nontrivial knot, by Proposition 2.5 it follows that $f_{10}(N_{10})$ also does not contain a nontrivial knot. Namely N_{10} is not IK. By repeating this argument, we can see that each of the graphs N_{11}, N'_{11} and N'_{12} is also not IK, see Fig. 3.1. \square

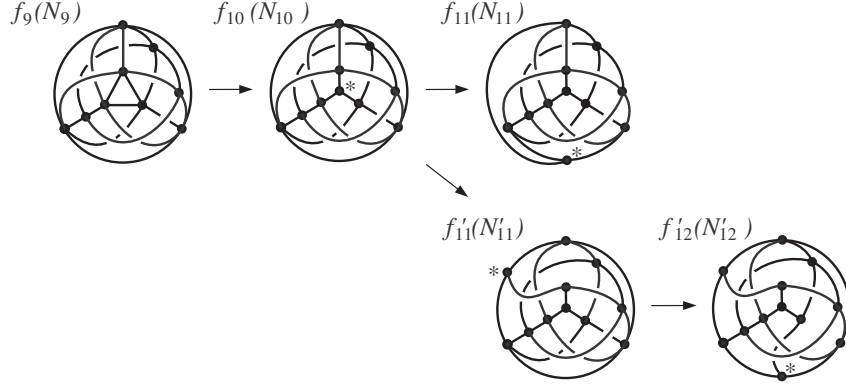


FIGURE 3.1.

Proof of Theorem 1.1. First we show that (1) and (2) are equivalent. Since we have already known that (2) implies (1), we show that (1) implies (2). If G is IK, then by Lemma 3.1 we see that G is not one of $N_9, N_{10}, N_{11}, N'_{10}, N'_{11}$ or N'_{12} . Namely G is obtained from K_7 by a finite sequence of ΔY -exchanges. Next we show that (2) and (3) are equivalent. Assume that G is obtained from K_7 by a finite sequence of ΔY -exchanges. Note that $\Gamma^{(3)}(K_7)$ is the empty set. Thus by Proposition 2.4 we see that $\Gamma^{(3)}(G)$ is the empty set. Conversely, if G is one of $N_9, N_{10}, N_{11}, N'_{10}, N'_{11}$, and N'_{12} , then $\Gamma^{(3)}(G)$ is not the empty set. This completes the proof. \square

Remark 3.2. Let f'_{11} be the spatial embedding of N'_{11} as illustrated in Fig. 3.1 and f'_{10} the spatial embedding of N'_{10} as illustrated in Fig. 3.2. Let $\varphi_{N'_{11}, N'_{10}}$ be the map from $\text{SE}(N'_{11})$ to $\text{SE}(N'_{10})$ induced by the $Y\Delta$ -exchange from N'_{11} to N'_{10} at the double-marked Y as illustrated in Fig. 3.2. Then it is clear that $\varphi(f'_{11}) = f'_{10}$. Moreover we can see that f'_{10} coincides with Flapan-Naimi's example of a spatial embedding of N'_{10} whose image does not contain a nontrivial knot as illustrated in the left side of Fig. 3.2 [3].

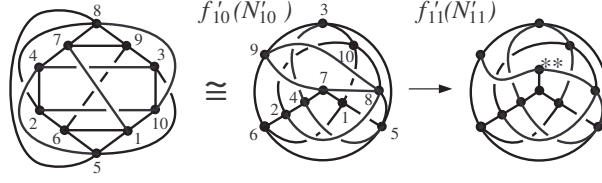


FIGURE 3.2.

4. PROOF OF THEOREM 1.2

We need some lemmas which are needed to prove Theorem 1.2.

Lemma 4.1. (Conway-Gordon [1], Taniyama-Yasuhara [21]) *Let G be a graph in the Petersen family and f a spatial embedding of G . Then there exists an element λ in $\Gamma^{(2)}(G)$ such that $\text{lk}(f(\lambda))$ is odd, where lk denotes the linking number in the 3-sphere.*

Let D_4 be the graph as illustrated in Fig. 4.1. We denote the set of all cycles with exactly four edges of D_4 by $\Gamma_4(D_4)$. For a spatial embedding f of D_4 , we define

$$\alpha(f) \equiv \sum_{\gamma \in \Gamma_4(D_4)} a_2(f(\gamma)) \pmod{2},$$

where a_2 denotes the second coefficient of the *Conway polynomial*. Note that $a_2(K)$ of a knot K is congruent to the *Arf invariant* modulo two [12]. Then the following is known.

Lemma 4.2. (Taniyama-Yasuhara [21]) *Let f be a spatial embedding of D_4 and λ, λ' all elements in $\Gamma^{(2)}(D_4)$. If both $\text{lk}(f(\lambda))$ and $\text{lk}(f(\lambda'))$ are odd, then $\alpha(f) = 1$.*

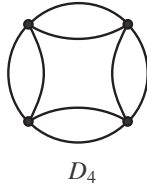


FIGURE 4.1.

Let G be a graph which contains D_4 as a minor and f a spatial embedding of G . Then we define

$$\alpha(f) \equiv \sum_{\gamma \in \Gamma_4(D_4)} a_2(f(\Psi_{D_4, G}(\gamma))) \pmod{2}.$$

Lemma 4.3. *Let G be a graph which contains D_4 as a minor and f a spatial embedding of G . For two elements μ and μ' in $\Psi_{D_4, G}^{(2)}(\Gamma^{(2)}(D_4))$, if both $\text{lk}(f(\mu))$ and $\text{lk}(f(\mu'))$ are odd, then $\alpha(f) = 1$.*

Proof. For two elements λ and λ' in $\Gamma^{(2)}(D_4)$, we see that both $\text{lk}(f(\Psi_{D_4,G}^{(2)}(\lambda)))$ and $\text{lk}(f(\Psi_{D_4,G}^{(2)}(\lambda')))$ are odd by the assumption. Then by Proposition 2.1, it follows that $\text{lk}(\psi_{G,D_4}(f)(\lambda))$ and $\text{lk}(\psi_{G,D_4}(f)(\lambda'))$ are also odd. Thus by Lemma 4.2, we have that

$$\begin{aligned} \alpha(f) &\equiv \sum_{\gamma \in \Gamma_4(D_4)} a_2(f(\Psi_{D_4,G}(\gamma))) \\ &= \sum_{\gamma \in \Gamma_4(D_4)} a_2(\psi_{G,D_4}(f)(\gamma)) \\ &\equiv 1 \pmod{2}. \end{aligned}$$

□

Now we show the following theorem, which is the most important part in the proof of Theorem 1.2.

Theorem 4.4. *Let G be N_9 or N'_{10} . For every spatial embedding f of G , there exists an element γ in $\Gamma(G)$ such that $a_2(f(\gamma))$ is odd, or there exists an element λ in $\Gamma^{(3)}(G)$ such that each 2-component sublink of $f(\lambda)$ has an odd linking number.*

Proof. We give a label to each vertex of G as illustrated in Fig. 4.2. In the following we denote a k -cycle $\overline{i_1 i_2} \cup \overline{i_2 i_3} \cup \cdots \cup \overline{i_{k-1} i_k} \cup \overline{i_k i_1}$ of G by $[i_1 i_2 \cdots i_k]$.

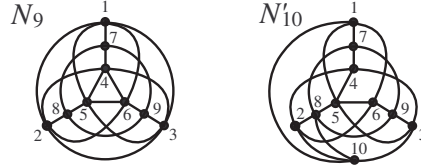


FIGURE 4.2.

First we show in the case of $G = N_9$. Let f be a spatial embedding of N_9 . Note that N_9 contains K_6 as the proper minor

$$(((N_9 - \overline{78}) - \overline{89}) - \overline{97}) / \overline{47/58/69}.$$

Thus by Lemma 4.1, there exists an element ν in $\Gamma^{(2)}(K_6)$ such that $\text{lk}(\psi_{N_9,K_6}(f)(\nu))$ is odd. Hence by Proposition 2.1, there exists an element μ in $\Psi_{K_6,N_9}^{(2)}(\Gamma^{(2)}(K_6))$ such that $\text{lk}(f(\mu))$ is odd. Note that $\Psi_{K_6,N_9}^{(2)}(\Gamma^{(2)}(K_6))$ consists of ten elements, and by the symmetry of N_9 , we may assume that $\mu = [1\ 7\ 4\ 3] \cup [2\ 6\ 5\ 8]$ or $[1\ 2\ 3] \cup [4\ 5\ 6]$ without loss of generality.

Case 1. $\mu = [1\ 7\ 4\ 3] \cup [2\ 6\ 5\ 8]$.

Note that N_9 contains P_7 as the proper minor

$$((((N_9 - \overline{61}) - \overline{62}) - \overline{64}) - \overline{65}) - \overline{69}) / \overline{39}.$$

Thus by Lemma 4.1, there exists an element ν' in $\Gamma^{(2)}(P_7)$ such that $\text{lk}(\psi_{N_9,P_7}(f)(\nu'))$ is odd. Hence by Proposition 2.1, there exists an element μ' in $\Psi_{P_7,N_9}^{(2)}(\Gamma^{(2)}(P_7))$

such that $\text{lk}(f(\mu'))$ is odd. Note that $\Psi_{P_7, N_9}^{(2)}(\Gamma^{(2)}(P_7))$ consists of nine elements

$$\begin{aligned}\mu'_1 &= [3\ 4\ 5] \cup [1\ 2\ 8\ 7], \mu'_2 = [1\ 5\ 4\ 7] \cup [2\ 3\ 9\ 8], \mu'_3 = [2\ 8\ 5\ 4] \cup [3\ 1\ 7\ 9], \\ \mu'_4 &= [1\ 2\ 4\ 7] \cup [3\ 5\ 8\ 9], \mu'_5 = [1\ 2\ 3] \cup [4\ 7\ 8\ 5], \mu'_6 = [1\ 2\ 8\ 5] \cup [3\ 4\ 7\ 9], \\ \mu'_7 &= [2\ 3\ 4] \cup [1\ 5\ 8\ 7], \mu'_8 = [7\ 8\ 9] \cup [1\ 2\ 4\ 5], \mu'_9 = [1\ 5\ 3] \cup [2\ 8\ 7\ 4].\end{aligned}$$

For $i = 1, 2, \dots, 9$, let J^i be the subgraph of N_9 which is $\mu \cup \mu'_i \cup \overline{6\ 9}$ if $i = 3, 6$ and $\mu \cup \mu'_i$ if $i \neq 3, 6$. Assume that $\text{lk}(f(\mu'_i))$ is odd for some $i \neq 8$. Then it can be easily seen that J^i contains a graph D^i as a minor so that D^i is isomorphic to D_4 and $\{\mu, \mu'_i\} = \Psi_{D^i, J^i}^{(2)}(\Gamma^{(2)}(D^i))$. Since both $\text{lk}(f(\mu))$ and $\text{lk}(f(\mu'_i))$ are odd, by Lemma 4.3 there exists an element γ in $\Gamma(J^i)$ such that $a_2(f(\gamma))$ is odd. Next assume that $\text{lk}(f(\mu'_8))$ is odd. We denote two elements $[7\ 8\ 9] \cup [1\ 2\ 6\ 5]$ and $[7\ 8\ 9] \cup [4\ 2\ 6\ 5]$ in $\Gamma^{(2)}(J^8)$ by $\mu'_{8,1}$ and $\mu'_{8,2}$, respectively. We denote the subgraph $\mu \cup \mu'_{8,j}$ of J^8 by $J^{8,j}$ ($j = 1, 2$). Then it can be easily seen that $J^{8,j}$ contains a graph $D^{8,j}$ as a minor so that $D^{8,j}$ is isomorphic to D_4 and $\{\mu, \mu'_{8,j}\} = \Psi_{D^{8,j}, J^{8,j}}^{(2)}(\Gamma^{(2)}(D^{8,j}))$ ($j = 1, 2$). Note that $[1\ 2\ 4\ 5] = [1\ 2\ 6\ 5] + [4\ 2\ 6\ 5]$ in $H_1(J^8; \mathbb{Z}_2)$, where $H_*(\cdot; \mathbb{Z}_2)$ denotes the homology group with \mathbb{Z}_2 -coefficients. Then, by the homological property of the linking number, we have that

$$1 \equiv \text{lk}(f(\mu'_8)) \equiv \text{lk}(f(\mu'_{8,1})) + \text{lk}(f(\mu'_{8,2})) \pmod{2}.$$

Thus we see that $\text{lk}(f(\mu'_{8,1}))$ is odd or $\text{lk}(f(\mu'_{8,2}))$ is odd. In either case, by Lemma 4.3 there exists an element γ in $\Gamma(J^{8,j})$ such that $a_2(f(\gamma))$ is odd.

Case 2. $\mu = [1\ 2\ 3] \cup [4\ 5\ 6]$.

Note that N_9 contains P_9 as the proper minor

$$((((N_9 - \overline{1\ 2}) - \overline{2\ 3}) - \overline{3\ 1}) - \overline{4\ 5}) - \overline{5\ 6}) - \overline{6\ 4}.$$

Thus by Lemma 4.1, there exists an element ν' in $\Gamma^{(2)}(P_9)$ such that $\text{lk}(\psi_{N_9, P_9}(f)(\nu'))$ is odd. Hence by Proposition 2.1, there exists an element μ' in $\Psi_{P_9, N_9}^{(2)}(\Gamma^{(2)}(P_9))$ such that $\text{lk}(f(\mu'))$ is odd. Note that $\Psi_{P_9, N_9}^{(2)}(\Gamma^{(2)}(P_9))$ consists of seven elements, and by the symmetry of N_9 , we may assume that $\mu' = [1\ 5\ 8\ 7] \cup [2\ 6\ 9\ 3\ 4]$ or $[7\ 8\ 9] \cup [1\ 5\ 3\ 4\ 2\ 6]$ without loss of generality. We denote the subgraph $\mu \cup \mu'$ of N_9 by J . Assume that $\mu' = [1\ 5\ 8\ 7] \cup [2\ 6\ 9\ 3\ 4]$. We denote two elements $[1\ 5\ 8\ 7] \cup [4\ 3\ 2]$ and $[1\ 5\ 8\ 7] \cup [6\ 9\ 3\ 2]$ in $\Gamma^{(2)}(J)$ by μ'_1 and μ'_2 , respectively. We denote the subgraph $\mu \cup \mu'_i$ of J by J^i ($i = 1, 2$). Then J^i contains a graph D^i as a minor so that D^i is isomorphic to D_4 and $\{\mu, \mu'_i\} = \Psi_{D^i, J^i}^{(2)}(\Gamma^{(2)}(D^i))$ ($i = 1, 2$). Since $[2\ 6\ 9\ 3\ 4] = [4\ 3\ 2] + [6\ 9\ 3\ 2]$ in $H_1(J; \mathbb{Z}_2)$, it follows that

$$1 \equiv \text{lk}(f(\mu')) \equiv \text{lk}(f(\mu'_1)) + \text{lk}(f(\mu'_2)) \pmod{2}.$$

This implies that $\text{lk}(f(\mu'_1))$ is odd or $\text{lk}(f(\mu'_2))$ is odd. In either case, by Lemma 4.3 there exists an element γ in $\Gamma(J^i)$ such that $a_2(f(\gamma))$ is odd. Next assume that $\mu' = [7\ 8\ 9] \cup [1\ 5\ 3\ 4\ 2\ 6]$. We denote four elements $[7\ 8\ 9] \cup [3\ 4\ 5]$, $[7\ 8\ 9] \cup [4\ 5\ 6]$, $[7\ 8\ 9] \cup [1\ 5\ 6]$ and $[7\ 8\ 9] \cup [2\ 4\ 6]$ in $\Gamma^{(2)}(J)$ by μ'_1, μ'_2, μ'_3 and μ'_4 , respectively. Since $[1\ 5\ 3\ 4\ 2\ 6] = [3\ 4\ 5] + [4\ 5\ 6] + [1\ 5\ 6] + [2\ 4\ 6]$ in $H_1(J; \mathbb{Z}_2)$, it follows that

$$1 \equiv \text{lk}(\mu') \equiv \text{lk}(\mu'_1) + \text{lk}(\mu'_2) + \text{lk}(\mu'_3) + \text{lk}(\mu'_4) \pmod{2}.$$

This implies that $\text{lk}(\mu'_i)$ is odd for some $i = 1, 2, 3$ or 4 . Moreover, by the symmetry of J , we may assume that $\text{lk}(\mu'_1)$ is odd or $\text{lk}(\mu'_2)$ is odd without loss of generality. Assume that $\text{lk}(\mu'_1)$ is odd. We denote the subgraph $\mu \cup \mu'_1 \cup \overline{1\ 7} \cup \overline{6\ 9}$ of N_9 by

J^1 . Then J^1 contains a graph D^1 as a minor so that D^1 is isomorphic to D_4 and $\{\mu, \mu'_1\} = \Psi_{D^1, J^1}^{(2)}(\Gamma^{(2)}(D^1))$. Since both $\text{lk}(f(\mu))$ and $\text{lk}(f(\mu'_1))$ are odd, by Lemma 4.3 there exists an element γ in $\Gamma(J^1)$ such that $a_2(f(\gamma))$ is odd. Next assume that $\text{lk}(\mu'_2)$ is odd. We denote four elements $[7\ 8\ 9] \cup [1\ 2\ 6]$, $[7\ 8\ 9] \cup [1\ 2\ 3]$, $[7\ 8\ 9] \cup [2\ 3\ 4]$ and $[7\ 8\ 9] \cup [1\ 3\ 5]$ in $\Gamma^{(2)}(J)$ by μ'_5, μ'_6, μ'_7 and μ'_8 , respectively. Since $[1\ 5\ 3\ 4\ 2\ 6] = [1\ 2\ 6] + [1\ 2\ 3] + [2\ 3\ 4] + [1\ 3\ 5]$ in $H_1(J; \mathbb{Z}_2)$, we have that

$$1 \equiv \text{lk}(\mu') \equiv \text{lk}(\mu'_5) + \text{lk}(\mu'_6) + \text{lk}(\mu'_7) + \text{lk}(\mu'_8) \pmod{2}.$$

Thus we see that $\text{lk}(\mu'_i)$ is odd for some $i = 5, 6, 7$ or 8 . Moreover, by the symmetry of J , we may assume that $\text{lk}(\mu'_5)$ is odd or $\text{lk}(\mu'_6)$ is odd without loss of generality. Assume that $\text{lk}(\mu'_5)$ is odd. We denote the subgraph $\mu \cup \mu'_5 \cup \overline{4\ 7} \cup \overline{3\ 9}$ of N_9 by J^5 . Then J^5 contains a graph D^5 as a minor so that D^5 is isomorphic to D_4 and $\{\mu, \mu'_5\} = \Psi_{D^5, J^5}^{(2)}(\Gamma^{(2)}(D^5))$. Since both $\text{lk}(f(\mu))$ and $\text{lk}(f(\mu'_5))$ are odd, by Lemma 4.3 there exists an element γ in $\Gamma(J^5)$ such that $a_2(f(\gamma))$ is odd. Finally, assume that $\text{lk}(\mu'_6)$ is odd. Let us consider the 3-component link $L = f([1\ 2\ 3] \cup [4\ 5\ 6] \cup [7\ 8\ 9])$. Since all 2-component sublinks of L are $f(\mu), f(\mu'_2)$ and $f(\mu'_6)$, each of the 2-component sublinks of L has an odd linking number.

Next we show in the case of $G = N'_{10}$. Let f be a spatial embedding of N'_{10} . Note that N'_{10} contains P_7 as the proper minor

$$(((N'_{10} - \overline{7\ 8}) - \overline{8\ 9}) - \overline{9\ 7}) / \overline{4\ 7} / \overline{5\ 8} / \overline{6\ 9}.$$

Thus by Lemma 4.1, there exists an element ν in $\Gamma^{(2)}(P_7)$ such that $\text{lk}(\psi_{N'_{10}, P_7}(f)(\nu))$ is odd. Hence by Proposition 2.1, there exists an element μ in $\Psi_{P_7, N'_{10}}^{(2)}(\Gamma^{(2)}(P_7))$ such that $\text{lk}(f(\mu))$ is odd. Note that $\Psi_{P_7, N'_{10}}^{(2)}(\Gamma^{(2)}(P_7))$ consists of nine elements, and by the symmetry of N'_{10} , we may assume that $\mu = [1\ 7\ 4\ 5] \cup [2\ 10\ 3\ 9\ 6]$, $[2\ 4\ 5\ 8] \cup [1\ 10\ 3\ 9\ 6]$, $[3\ 10\ 8\ 5] \cup [1\ 6\ 2\ 4\ 7]$, $[3\ 4\ 5] \cup [1\ 10\ 2\ 6]$ or $[2\ 8\ 10] \cup [1\ 6\ 9\ 3\ 4\ 7]$ without loss of generality.

Case 1. $\mu = [1\ 7\ 4\ 5] \cup [2\ 10\ 3\ 9\ 6]$.

Note that N'_{10} contains P_9 as the proper minor

$$((((N'_{10} - \overline{5\ 1}) - \overline{5\ 3}) - \overline{5\ 4}) - \overline{5\ 6}) - \overline{5\ 8}) - \overline{7\ 9}.$$

Thus by Lemma 4.1, there exists an element ν' in $\Gamma^{(2)}(P_9)$ such that $\text{lk}(\psi_{N'_{10}, P_9}(f)(\nu'))$ is odd. Hence by Proposition 2.1, there exists an element μ' in $\Psi_{P_9, N'_{10}}^{(2)}(\Gamma^{(2)}(P_9))$ such that $\text{lk}(f(\mu'))$ is odd. Note that $\Psi_{P_9, N'_{10}}^{(2)}(\Gamma^{(2)}(P_9))$ consists of seven elements

$$\begin{aligned} \mu'_1 &= [3\ 10\ 8\ 9] \cup [1\ 6\ 2\ 4\ 7], \quad \mu'_2 = [1\ 7\ 8\ 10] \cup [2\ 4\ 3\ 9\ 6], \\ \mu'_3 &= [1\ 10\ 2\ 6] \cup [3\ 4\ 7\ 8\ 9], \quad \mu'_4 = [2\ 4\ 3\ 10] \cup [1\ 7\ 8\ 9\ 6], \\ \mu'_5 &= [2\ 4\ 7\ 8] \cup [1\ 10\ 3\ 9\ 6], \quad \mu'_6 = [2\ 8\ 9\ 6] \cup [1\ 10\ 3\ 4\ 7], \\ \mu'_7 &= [2\ 8\ 10] \cup [1\ 6\ 9\ 3\ 4\ 7]. \end{aligned}$$

For $i = 1, 2, \dots, 7$, let J^i be the subgraph of N'_{10} which is $\mu \cup \mu'_i \cup \overline{5\ 8}$ if $i = 1, 6, 7$ and $\mu \cup \mu'_i$ if $i = 2, 3, 4, 5$. Assume that $\text{lk}(f(\mu'_i))$ is odd for some i . Then J^i contains a graph D^i as a minor so that D^i is isomorphic to D_4 and $\{\mu, \mu'_i\} = \Psi_{D^i, J^i}^{(2)}(\Gamma^{(2)}(D^i))$. Since both $\text{lk}(f(\mu))$ and $\text{lk}(f(\mu'_i))$ are odd, by Lemma 4.3 there exists an element γ in $\Gamma(J^i)$ such that $a_2(f(\gamma))$ is odd.

Case 2. $\mu = [2\ 4\ 5\ 8] \cup [1\ 10\ 3\ 9\ 6]$.

Note that N'_{10} contains another P_9 as the proper minor

$$(((N'_{10} - \overline{8\ 2}) - \overline{8\ 5}) - \overline{8\ 7}) - \overline{8\ 9}) - \overline{8\ 10}) - \overline{3\ 4}.$$

Thus by Lemma 4.1, there exists an element ν' in $\Gamma^{(2)}(P_9)$ such that $\text{lk}(\psi_{N'_{10}, P_9}(f)(\nu'))$ is odd. Hence by Proposition 2.1, there exists an element μ' in $\Psi_{P_9, N'_{10}}^{(2)}(\Gamma^{(2)}(P_9))$ such that $\text{lk}(f(\mu'))$ is odd. Note that $\Psi_{P_9, N'_{10}}^{(2)}(\Gamma^{(2)}(P_9))$ consists of seven elements

$$\begin{aligned}\mu'_1 &= [1\ 6\ 9\ 7] \cup [2\ 4\ 5\ 3\ 10], \quad \mu'_2 = [1\ 7\ 4\ 5] \cup [2\ 10\ 3\ 9\ 6], \\ \mu'_3 &= [3\ 5\ 6\ 9] \cup [1\ 10\ 2\ 4\ 7], \quad \mu'_4 = [1\ 5\ 3\ 10] \cup [2\ 4\ 7\ 9\ 6], \\ \mu'_5 &= [1\ 10\ 2\ 6] \cup [3\ 9\ 7\ 4\ 5], \quad \mu'_6 = [1\ 5\ 6] \cup [2\ 4\ 7\ 9\ 3\ 10], \\ \mu'_7 &= [2\ 4\ 5\ 6] \cup [1\ 10\ 3\ 9\ 7].\end{aligned}$$

For $i = 1, 2, \dots, 7$, let J^i be the subgraph of N'_{10} which is $\mu \cup \mu'_i \cup \overline{7\ 8}$ if $i = 1, 7$ and $\mu \cup \mu'_i$ if $i \neq 1, 7$. Assume that $\text{lk}(f(\mu'_i))$ is odd for some i . Then J^i contains a graph D^i as a minor so that D^i is isomorphic to D_4 and $\{\mu, \mu'_i\} = \Psi_{D^i, J^i}^{(2)}(\Gamma^{(2)}(D^i))$. Since both $\text{lk}(f(\mu))$ and $\text{lk}(f(\mu'_i))$ are odd, by Lemma 4.3 there exists an element γ in $\Gamma(J^i)$ such that $a_2(f(\gamma))$ is odd.

Case 3. $\mu = [3\ 10\ 8\ 5] \cup [1\ 6\ 2\ 4\ 7]$.

Let P_9 be the proper minor of N'_{10} and μ'_i the element in $\Psi_{P_9, N'_{10}}^{(2)}(\Gamma^{(2)}(P_9))$ ($i = 1, 2, \dots, 7$) as in Case 2. For $i = 1, 2, \dots, 7$, let J^i be the subgraph of N'_{10} which is $\mu \cup \mu'_i \cup \overline{8\ 9}$ if $i = 1, 4$ and $\mu \cup \mu'_i$ if $i \neq 1, 4$. Assume that $\text{lk}(f(\mu'_i))$ is odd for some i . Then J^i contains a graph D^i as a minor so that D^i is isomorphic to D_4 and $\{\mu, \mu'_i\} = \Psi_{D^i, J^i}^{(2)}(\Gamma^{(2)}(D^i))$. Since both $\text{lk}(f(\mu))$ and $\text{lk}(f(\mu'_i))$ are odd, by Lemma 4.3 there exists an element γ in $\Gamma(J^i)$ such that $a_2(f(\gamma))$ is odd.

Case 4. $\mu = [3\ 4\ 5] \cup [1\ 10\ 2\ 6]$.

Note that N'_{10} contains another P_7 as the proper minor

$$(((N'_{10} - \overline{3\ 4}) - \overline{4\ 5}) - \overline{5\ 3}) / \overline{3\ 9\ 4\ 7\ 5\ 8}.$$

Thus by Lemma 4.1, there exists an element ν' in $\Gamma^{(2)}(P_7)$ such that $\text{lk}(\psi_{N'_{10}, P_7}(f)(\nu'))$ is odd. Hence by Proposition 2.1, there exists an element μ' in $\Psi_{P_7, N'_{10}}^{(2)}(\Gamma^{(2)}(P_7))$ such that $\text{lk}(f(\mu'))$ is odd. Note that $\Psi_{P_7, N'_{10}}^{(2)}(\Gamma^{(2)}(P_7))$ consists of nine elements

$$\begin{aligned}\mu'_1 &= [5\ 6\ 9\ 8] \cup [1\ 10\ 2\ 4\ 7], \quad \mu'_2 = [3\ 10\ 8\ 9] \cup [1\ 6\ 2\ 4\ 7], \\ \mu'_3 &= [1\ 5\ 8\ 10] \cup [2\ 4\ 7\ 9\ 6], \quad \mu'_4 = [7\ 8\ 9] \cup [1\ 10\ 2\ 6], \\ \mu'_5 &= [2\ 8\ 10] \cup [1\ 6\ 9\ 7], \quad \mu'_6 = [2\ 8\ 5\ 6] \cup [1\ 10\ 3\ 9\ 7], \\ \mu'_7 &= [1\ 7\ 8\ 5] \cup [2\ 10\ 3\ 9\ 6], \quad \mu'_8 = [1\ 5\ 6] \cup [2\ 4\ 7\ 9\ 3\ 10], \\ \mu'_9 &= [2\ 4\ 7\ 8] \cup [1\ 10\ 3\ 9\ 6].\end{aligned}$$

For $i = 1, 2, \dots, 9$, let J^i be the subgraph of N'_{10} which is $\mu \cup \mu'_5 \cup \overline{4\ 7} \cup \overline{5\ 8}$ if $i = 5$ and $\mu \cup \mu'_i$ if $i \neq 5$. Assume that $\text{lk}(f(\mu'_i))$ is odd for some $i \neq 4, 8$. Then J^i contains a graph D^i as a minor so that D^i is isomorphic to D_4 and $\{\mu, \mu'_i\} = \Psi_{D^i, J^i}^{(2)}(\Gamma^{(2)}(D^i))$. Since both $\text{lk}(f(\mu))$ and $\text{lk}(f(\mu'_i))$ are odd, by Lemma 4.3 there exists an element γ in $\Gamma(J^i)$ such that $a_2(f(\gamma))$ is odd. Next assume that $\text{lk}(f(\mu'_8))$ is odd. We denote two elements $[1\ 5\ 6] \cup [2\ 4\ 3\ 10]$ and $[1\ 5\ 6] \cup [3\ 4\ 7\ 9]$ in $\Gamma^{(2)}(J^8)$ by $\mu'_{8,1}$ and $\mu'_{8,2}$, respectively. We denote the subgraph $\mu \cup \mu'_{8,1}$ of J^8 by $J^{8,1}$ and the subgraph

$\mu \cup \mu'_{8,2} \cup \overline{8\ 9} \cup \overline{8\ 10}$ of N'_{10} by $J^{8,2}$. Then $J^{8,j}$ contains a graph $D^{8,j}$ as a minor so that $D^{8,j}$ is isomorphic to D_4 and $\{\mu, \mu'_{8,j}\} = \Psi_{D^{8,j}, J^{8,j}}^{(2)}(\Gamma^{(2)}(D^{8,j}))$ ($j = 1, 2$). Since $[2\ 4\ 7\ 9\ 3\ 10] = [2\ 4\ 3\ 10] + [3\ 4\ 7\ 9]$ in $H_1(J^8; \mathbb{Z}_2)$, it follows that

$$1 \equiv \text{lk}(f(\mu'_8)) \equiv \text{lk}(f(\mu'_{8,1})) + \text{lk}(f(\mu'_{8,2})) \pmod{2}.$$

This implies that $\text{lk}(f(\mu'_{8,1}))$ is odd or $\text{lk}(f(\mu'_{8,2}))$ is odd. In either case, by Lemma 4.3 there exists an element γ in $\Gamma(J^{8,j})$ such that $a_2(f(\gamma))$ is odd. Finally assume that $\text{lk}(f(\mu'_4))$ is odd. Note that N'_{10} contains another P_9 as the proper minor

$$((((N'_{10} - \overline{2\ 4}) - \overline{2\ 6}) - \overline{2\ 8}) - \overline{2\ 10}) - \overline{5\ 1}) - \overline{5\ 3}.$$

Thus by Lemma 4.1, there exists an element ν'' in $\Gamma^{(2)}(P_9)$ such that $\text{lk}(\psi_{N'_{10}, P_9}(f)(\nu''))$ is odd. Hence by Proposition 2.1, there exists an element μ'' in $\Psi_{P_9, N'_{10}}^{(2)}(\Gamma^{(2)}(P_9))$ such that $\text{lk}(f(\mu''))$ is odd. Note that $\Psi_{P_9, N'_{10}}^{(2)}(\Gamma^{(2)}(P_9))$ consists of seven elements

$$\begin{aligned} \mu''_1 &= [5\ 6\ 9\ 8] \cup [1\ 10\ 3\ 4\ 7], \quad \mu''_2 = [4\ 5\ 8\ 7] \cup [1\ 10\ 3\ 9\ 6], \\ \mu''_3 &= [1\ 7\ 8\ 10] \cup [3\ 4\ 5\ 6\ 9], \quad \mu''_4 = [3\ 10\ 8\ 9] \cup [1\ 7\ 4\ 5\ 6], \\ \mu''_5 &= [1\ 6\ 9\ 7] \cup [3\ 4\ 5\ 8\ 10], \quad \mu''_6 = [3\ 9\ 7\ 4] \cup [1\ 10\ 8\ 5\ 6], \\ \mu''_7 &= [7\ 8\ 9] \cup [1\ 10\ 3\ 4\ 5\ 6]. \end{aligned}$$

For $j = 1, 2, \dots, 7$, let $J^{4,j}$ be the subgraph of N'_{10} which is $\mu'_4 \cup \mu''_j \cup \overline{2\ 4}$ if $j = 2, 6$ and $\mu'_4 \cup \mu''_j$ if $j \neq 2, 6$. Assume that $\text{lk}(f(\mu''_j))$ is odd for some $j \neq 7$. Then $J^{4,j}$ contains a graph $D^{4,j}$ as a minor so that $D^{4,j}$ is isomorphic to D_4 and $\{\mu'_4, \mu''_j\} = \Psi_{D^{4,j}, J^{4,j}}^{(2)}(\Gamma^{(2)}(D^{4,j}))$. Since both $\text{lk}(f(\mu'_4))$ and $\text{lk}(f(\mu''_j))$ are odd, by Lemma 4.3 there exists an element γ in $\Gamma(J^{4,j})$ such that $a_2(f(\gamma))$ is odd. Next assume that $\text{lk}(f(\mu''_7))$ is odd. We denote three elements $[7\ 8\ 9] \cup [1\ 5\ 3\ 10]$, $[7\ 8\ 9] \cup [1\ 5\ 6]$ and $[7\ 8\ 9] \cup [3\ 4\ 5]$ in $\Gamma^{(2)}(N'_{10})$ by $\mu''_{7,1}$, $\mu''_{7,2}$ and $\mu''_{7,3}$, respectively. We denote the subgraph $\mu \cup \mu''_{7,k} \cup \overline{4\ 7} \cup \overline{2\ 8}$ of N'_{10} by $J^{4,7,k}$ ($k = 1, 2$). Then $J^{4,7,k}$ contains a graph $D^{4,7,k}$ as a minor so that $D^{4,7,k}$ is isomorphic to D_4 and $\{\mu, \mu''_{7,k}\} = \Psi_{D^{4,7,k}, J^{4,7,k}}^{(2)}(\Gamma^{(2)}(D^{4,7,k}))$ ($k = 1, 2$). Since $[1\ 10\ 3\ 4\ 5\ 6] = [1\ 5\ 3\ 10] + [1\ 5\ 6] + [3\ 4\ 5]$ in $H_1(N'_{10}; \mathbb{Z}_2)$, it follows that

$$1 \equiv \text{lk}(f(\mu'_7)) \equiv \text{lk}(f(\mu''_{7,1})) + \text{lk}(f(\mu''_{7,2})) + \text{lk}(f(\mu''_{7,3})) \pmod{2}.$$

This implies that $\text{lk}(f(\mu''_{7,k}))$ is odd for some k . If $\text{lk}(f(\mu''_{7,1}))$ is odd or $\text{lk}(f(\mu''_{7,2}))$ is odd, then by Lemma 4.3 there exists an element γ in $\Gamma(J^{4,7,k})$ such that $a_2(f(\gamma))$ is odd. If $\text{lk}(f(\mu''_{7,3}))$ is odd, let us consider the 3-component link $L = f([3\ 4\ 5] \cup [7\ 8\ 9] \cup [1\ 10\ 2\ 6])$. Since all 2-component sublinks of L are $f(\mu)$, $f(\mu'_4)$ and $f(\mu''_{7,3})$, each of the 2-component sublinks of L has an odd linking number.

Case 5. $\mu = [2\ 8\ 10] \cup [1\ 6\ 9\ 3\ 4\ 7]$.

We denote two elements $[2\ 8\ 10] \cup [1\ 6\ 9\ 7]$ and $[2\ 8\ 10] \cup [3\ 9\ 7\ 4]$ in $\Gamma^{(2)}(N'_{10})$ by μ_1 and μ_2 , respectively. Since $[1\ 6\ 9\ 3\ 4\ 7] = [1\ 6\ 9\ 7] + [3\ 9\ 7\ 4]$ in $H_1(N'_{10}; \mathbb{Z}_2)$, it follows that

$$1 \equiv \text{lk}(f(\mu)) \equiv \text{lk}(f(\mu_1)) + \text{lk}(f(\mu_2)) \pmod{2}.$$

This implies that $\text{lk}(f(\mu_1))$ is odd or $\text{lk}(f(\mu_2))$ is odd. By the symmetry of N'_{10} , we may assume that $\text{lk}(f(\mu_1))$ is odd. Note that N'_{10} contains another P_7 as the

proper minor

$$(((N'_{10} - \overline{2\ 8}) - \overline{8\ 10}) - \overline{10\ 2}) / \overline{2\ 6/3\ 10/5\ 8}.$$

Thus by Lemma 4.1, there exists an element ν' in $\Gamma^{(2)}(P_7)$ such that $\text{lk}(\psi_{N'_{10}, P_7}(f)(\nu'))$ is odd. Hence by Proposition 2.1, there exists an element μ' in $\Psi_{P_7, N'_{10}}^{(2)}(\Gamma^{(2)}(P_7))$ such that $\text{lk}(f(\mu'))$ is odd. Note that $\Psi_{P_7, N'_{10}}^{(2)}(\Gamma^{(2)}(P_7))$ consists of nine elements

$$\begin{aligned}\mu'_1 &= [3\ 5\ 8\ 9] \cup [1\ 6\ 2\ 4\ 7], \quad \mu'_2 = [1\ 7\ 8\ 5] \cup [2\ 4\ 3\ 9\ 6], \\ \mu'_3 &= [1\ 5\ 6] \cup [3\ 9\ 7\ 4], \quad \mu'_4 = [3\ 4\ 5] \cup [1\ 6\ 9\ 7], \\ \mu'_5 &= [5\ 6\ 9\ 8] \cup [1\ 10\ 3\ 4\ 7], \quad \mu'_6 = [4\ 5\ 8\ 7] \cup [1\ 10\ 3\ 9\ 6], \\ \mu'_7 &= [1\ 5\ 3\ 10] \cup [2\ 4\ 7\ 9\ 6], \quad \mu'_8 = [2\ 4\ 5\ 6] \cup [1\ 10\ 3\ 9\ 7], \\ \mu'_9 &= [7\ 8\ 9] \cup [1\ 10\ 3\ 4\ 2\ 6].\end{aligned}$$

For $i = 1, 2, \dots, 9$, let J^i be the subgraph of N'_{10} which is $\mu_1 \cup \mu'_3 \cup \overline{3\ 10} \cup \overline{5\ 8}$ if $i = 3$ and $\mu_1 \cup \mu'_i$ if $i \neq 3$. Assume that $\text{lk}(f(\mu'_i))$ is odd for some $i \neq 4, 9$. Then J^i contains a graph D^i as a minor so that D^i is isomorphic to D_4 and $\{\mu_1, \mu'_i\} = \Psi_{D^i, J^i}^{(2)}(\Gamma^{(2)}(D^i))$. Since both $\text{lk}(f(\mu_1))$ and $\text{lk}(f(\mu'_i))$ are odd, by Lemma 4.3 there exists an element γ in $\Gamma(J^i)$ such that $a_2(f(\gamma))$ is odd. Next assume that $\text{lk}(f(\mu'_9))$ is odd. We denote two elements $[7\ 8\ 9] \cup [1\ 6\ 2\ 10]$ and $[7\ 8\ 9] \cup [2\ 4\ 3\ 10]$ in $\Gamma^{(2)}(J^9)$ by $\mu'_{9,1}$ and $\mu'_{9,2}$, respectively. We denote the subgraph $\mu_1 \cup \mu'_{8,1}$ of J^9 by $J^{9,1}$ and the subgraph $\mu_1 \cup \mu'_{9,2} \cup \overline{5\ 3} \cup \overline{5\ 1}$ of N'_{10} by $J^{9,2}$. Then $J^{9,j}$ contains a graph $D^{9,j}$ as a minor so that $D^{9,j}$ is isomorphic to D_4 and $\{\mu_1, \mu'_{9,j}\} = \Psi_{D^{9,j}, J^{9,j}}^{(2)}(\Gamma^{(2)}(D^{9,j}))$ ($j = 1, 2$). Since $[1\ 10\ 3\ 4\ 2\ 6] = [1\ 6\ 2\ 10] + [2\ 4\ 3\ 10]$ in $H_1(J^9; \mathbb{Z}_2)$, it follows that

$$1 \equiv \text{lk}(f(\mu'_9)) \equiv \text{lk}(f(\mu'_{9,1})) + \text{lk}(f(\mu'_{9,2})) \pmod{2}.$$

This implies that $\text{lk}(f(\mu'_{9,1}))$ is odd or $\text{lk}(f(\mu'_{9,2}))$ is odd. In either case, by Lemma 4.3 there exists an element γ in $\Gamma(J^{9,j})$ such that $a_2(f(\gamma))$ is odd. Finally assume that $\text{lk}(f(\mu'_4))$ is odd. Note that N'_{10} contains another P_9 as the proper minor

$$((((N'_{10} - \overline{6\ 1}) - \overline{6\ 2}) - \overline{6\ 5}) - \overline{6\ 9}) - \overline{8\ 7}) - \overline{8\ 10}.$$

Thus by Lemma 4.1, there exists an element ν'' in $\Gamma^{(2)}(P_9)$ such that $\text{lk}(\psi_{N'_{10}, P_9}(f)(\nu''))$ is odd. Hence by Proposition 2.1, there exists an element μ'' in $\Psi_{P_9, N'_{10}}^{(2)}(\Gamma^{(2)}(P_9))$ such that $\text{lk}(f(\mu''))$ is odd. Note that $\Psi_{P_9, N'_{10}}^{(2)}(\Gamma^{(2)}(P_9))$ consists of seven elements

$$\begin{aligned}\mu''_1 &= [3\ 5\ 8\ 9] \cup [1\ 10\ 2\ 4\ 7], \quad \mu''_2 = [3\ 9\ 7\ 4] \cup [1\ 5\ 8\ 2\ 10], \\ \mu''_3 &= [1\ 7\ 4\ 5] \cup [2\ 8\ 9\ 3\ 10], \quad \mu''_4 = [2\ 4\ 5\ 8] \cup [1\ 10\ 3\ 9\ 7], \\ \mu''_5 &= [2\ 4\ 3\ 10] \cup [1\ 5\ 8\ 9\ 7], \quad \mu''_6 = [1\ 5\ 3\ 10] \cup [2\ 4\ 7\ 9\ 8], \\ \mu''_7 &= [3\ 4\ 5] \cup [1\ 10\ 2\ 8\ 9\ 7].\end{aligned}$$

For $j = 1, 2, \dots, 7$, let $J^{4,j}$ be the subgraph of N'_{10} which is $\mu'_4 \cup \mu''_j \cup \overline{2\ 6}$ if $j = 4, 5$ and $\mu'_4 \cup \mu''_j$ if $j \neq 4, 5$. Assume that $\text{lk}(f(\mu''_j))$ is odd for some $j \neq 7$. Then $J^{4,j}$ contains a graph $D^{4,j}$ as a minor so that $D^{4,j}$ is isomorphic to D_4 and $\{\mu'_4, \mu''_j\} = \Psi_{D^{4,j}, J^{4,j}}^{(2)}(\Gamma^{(2)}(D^{4,j}))$. Since both $\text{lk}(f(\mu'_4))$ and $\text{lk}(f(\mu''_j))$ are odd, by Lemma 4.3 there exists an element γ in $\Gamma(J^{4,j})$ such that $a_2(f(\gamma))$ is odd. Next assume that $\text{lk}(f(\mu''_7))$ is odd. We denote two elements $[3\ 4\ 5] \cup [1\ 10\ 8\ 9\ 7]$ and $[3\ 4\ 5] \cup [2\ 8\ 10]$ in $\Gamma^{(2)}(N'_{10})$ by $\mu''_{7,1}$ and $\mu''_{7,2}$, respectively. We denote the subgraph $\mu_1 \cup \mu''_{7,1} \cup \overline{2\ 4} \cup \overline{5\ 6}$ of N'_{10} by $J^{4,7}$. Then $J^{4,7}$ contains a graph $D^{4,7}$ as a minor so that $D^{4,7}$

is isomorphic to D_4 and $\{\mu_1, \mu''_{7,1}\} = \Psi_{D^{4,7}, J^{4,7}}^{(2)}(\Gamma^{(2)}(D^{4,7}))$. Since $[1\ 10\ 2\ 8\ 9\ 7] = [1\ 10\ 8\ 9\ 7] + [2\ 8\ 10]$ in $H_1(N'_{10}; \mathbb{Z}_2)$, it follows that

$$1 \equiv \text{lk}(f(\mu'_7)) \equiv \text{lk}(f(\mu''_{7,1})) + \text{lk}(f(\mu''_{7,2})) \pmod{2}.$$

This implies that $\text{lk}(f(\mu''_{7,1}))$ is odd or $\text{lk}(f(\mu''_{7,2}))$ is odd. If $\text{lk}(f(\mu''_{7,1}))$ is odd, then by Lemma 4.3 there exists an element γ in $\Gamma(J^{4,7})$ such that $a_2(f(\gamma))$ is odd. If $\text{lk}(f(\mu''_{7,2}))$ is odd, let us consider the 3-component link $L = f([3\ 4\ 5] \cup [2\ 8\ 10] \cup [1\ 6\ 9\ 7])$. Since all 2-component sublinks of L are $f(\mu_1)$, $f(\mu'_4)$ and $f(\mu''_{7,2})$, each of the 2-component sublinks of L has an odd linking number. This completes the proof. \square

Proof of Theorem 1.2. Note that a graph in the Heawood family is obtained from one of K_7 , N_9 and N'_{10} by a finite sequence of $\triangle Y$ -exchanges. Thus by Lemma 2.6, Theorem 4.4 and the fact that K_7 is IK, namely I(K or C3L), it follows that every graph in the Heawood family is I(K or C3L). On the other hand, a graph in the Heawood family is obtained from one of H_{12} and C_{14} by a finite sequence of $Y\triangle$ -exchanges. Since each of H_{12} and C_{14} is a minor-minimal IK graph and $\Gamma^{(3)}(H_{12})$ and $\Gamma^{(3)}(C_{14})$ are the empty sets, it follows that H_{12} and C_{14} are minor-minimal I(K or C3L) graphs. Thus by Lemma 2.7, we have the desired conclusion. \square

Remark 4.5. A graph is said to be 2-apex if it can be embedded in the 2-sphere after the deletion of at most two vertices and all of their incidental edges. It is not hard to see that any 2-apex graph may have a spatial embedding whose image contains neither a nontrivial knot nor a 3-component link any of whose 2-component sublink is nonsplittable. Thus any 2-apex graph is not I(K or C3L). It is known that every graph of at most twenty edges is 2-apex [14] (see also [11]). Since the number of all edges of every graph in the Heawood family is twenty one, we see that any proper minor of a graph in the Heawood family is 2-apex, namely not I(K or C3L). This also implies that any graph in the Heawood family is minor-minimal for I(K or C3L).

Example 4.6. Let g_9 be the spatial embedding of N_9 and g'_{10} the spatial embedding of N'_{10} as illustrated in Fig. 4.3. Then it can be checked directly that both $g_9(N_9)$ and $g'_{10}(N'_{10})$ do not contain a nonsplittable 3-component link. Thus neither N_9 nor N'_{10} is I3L. Moreover, we can see that N_{10} , N_{11} , N'_{11} and N'_{12} are not I3L in a similar way as the proof of Lemma 3.1, see Fig. 4.3.

Remark 4.7. We remark that the Heawood graph is IK. The Heawood graph is the dual graph of K_7 which is embedded in a torus. It is known that there exists a unique graph C_{14} obtained from K_7 by seven times applications of $\triangle Y$ -exchanges [13]. The seven triangles corresponds to the black triangles of black-and-white coloring of the torus by K_7 . Then C_{14} and H are mapped to each other by parallel transformation of the torus, see Fig. 4.4. Thus they are isomorphic. Since C_{14} is IK, we have the result.

Remark 4.8. It is known that all twenty six graphs obtained from the complete four-partite graph $K_{3,3,1,1}$ by a finite sequence of $\triangle Y$ -exchanges are minor-minimal IK graphs [13], [6]. There exist thirty two graphs which are obtained from $K_{3,3,1,1}$ by a finite sequence of $\triangle Y$ and $Y\triangle$ -exchanges but cannot be obtained from $K_{3,3,1,1}$ by a finite sequence of $\triangle Y$ -exchanges. Recently, Goldberg-Mattman-Naimi announces that these thirty two graphs are also minor-minimal IK graphs [9].

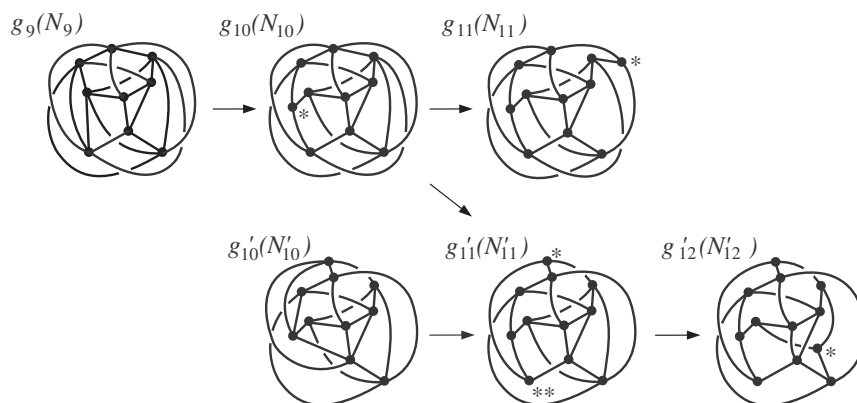


FIGURE 4.3.

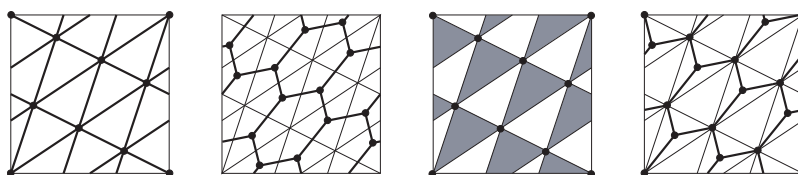


FIGURE 4.4.

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